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PLANE DYNAMIC PROBLEMS FOR ELASTIC INCOMPRESSIBLE BODIES WITH INITIAL STRESSES*

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Plane dynamic problems for elastic incompressible bodies with initial stresses are considered when the original dynamic problems admit reduction to steady state problems in a system of coordinates in rectilinear motion at constant velocity. The investigation is carried out using relations of the linearized theory of elasticity. Complex potentials are introduced for equal and unequal roots of the controlling equation with potentials of arbitrary form. A method of obtaining exact solutions using complex potentials is indicated in the case of bodies with initial stresses. Values of complex parameters are presented for bodies with elastic potential of specific form. A number of similar results was obtained in /1/ for compressible bodies with initial stresses.

Plane dynamic problems of the classical theory of elasticity have an exact solution when the original dynamic problem admits reduction to the steady state problem in a system of coordinates moving rectilinearly at constant velocity. Beginning with /2/ these problems were considered in the case of an isotropic /2 – 8/ and orthotropic /9/ bodies. These problems were solved in /2 – 4, 7 – 9/ using the representation of stresses and displacements in terms of complex potentials, which facilitated the derivation of solutions in closed form. Complex potentials were used in the case of plane static problems for compressible and incompressible elastic bodies with homogeneous initial stresses in /10,11/ and /12/, respectively, for unequal and equal roots. Relations of the linearized theory of elasticity were used in /13 – 15/ for small and large initial deformations in the case of bodies with the elastic potential of arbitrary form. The complex potentials introduced in /10 – 12/ in the absence of initial stresses become in the case of unequal roots Lekhnitskii's complex potentials /16/ for an orthotropic linear elastic body, and in the case of equal roots they become the complex potentials of Kolosov – Muskhelishvili /17/ for the isotropic linear elastic body.

1. Statement of the problem. Basic relations. Let us consider a nonlinearly elastic isotropic incompressible body with the elastic potential of arbitrary form. The results presented below apply also to an orthotropic incompressible body when the equivalent elastic directions coincide with those of coordinate lines of the selected coordinate system. We introduce the following coordinates: Lagrangian coordinates x_j which in the natural (undeformed) state are the same as Cartesian coordinates; Cartesian coordinates η_j of the system moving rectilinearly along the Oy_1 at constant velocity v; Cartesian coordinates for the initial (deformed state) y_j. All quantities related to the initial state are denoted by a zero superscript; analysis is carried out using the linearized theory of elasticity /13 – 15/ in the form that is general in the theory of finite (large) initial deformations and all variants of the theory of small initial deformations. The problem becomes specifically defined by the selection of expressions for the coefficients in basic equations /15/. We assume the initial conditions to be homogeneous, so that displacements can be expressed in the form

$$u_n{}^0 = \delta_{nm} (\lambda_m - 1) x_m;\ \lambda_m = \mathrm{const};\ n,\ m = 1,\ 2 \tag{1.1}$$

where λ_m are the coefficients of elongation along the coordinate axes and δ_{nm} is the Kronecker delta. In conformity with the above statement and (1.1) the normal stresses $S_{11}{}^\circ \equiv \sigma_{11}{}^{*\circ} \neq 0$; $S_{22} \equiv \sigma_{22}{}^{*\circ} \neq 0$ are generally the nonzero stress tensor components in the initial state /13 – 15/.

Under condition (1.1) the coordinates of the introduced systems are connected by the relationships

$$y_j = \lambda_j x_j;\ \eta_1 = y_1 - vt;\ \eta_2 \equiv y_2;\ \ j = 1,\ 2;\ \ v = \mathrm{const} \tag{1.2}$$

By virtue of the incompressibility condition the elongation coefficients are linked in the initial state by the relation

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$$\lambda_1\lambda_2\lambda_3 = 1 \tag{1.3}$$

Below, all perturbations are related to dimensions of the body in the initial deformed state; because of this the perturbation components Q_{nm}^* of the stress tensor are related to the area of the surface on which they are acting, and by virtue of relations (1.3) the density ρ of material will be assumed related to both the natural and the initial deformed state.

Let us consider the basic relationships of the plane dynamic problem of the theory of elasticity for compressible bodies with initial stresses in the formulation of /13–15/. According to /10,11/ these relationships reduce to the equations of motion

$$\begin{aligned} &\frac{\partial}{\partial y_1} Q_{11}^* + \frac{\partial}{\partial y_2} Q_{21}^* - \rho \frac{\partial^2}{\partial t^2} u_1 = 0; \\ &\frac{\partial}{\partial y_1} Q_{12}^* + \frac{\partial}{\partial y_2} Q_{22}^* - \rho \frac{\partial^2}{\partial t^2} u_2 = 0 \end{aligned} \tag{1.4}$$

with the incompressibility condition

$$\lambda_1 q_1 \frac{\partial}{\partial y_1} u_1 + \lambda_2 q_2 \frac{\partial}{\partial y_2} u_2 = 0 \tag{1.5}$$

and elasticity relations of the form

$$\begin{aligned} Q_{11}^* &= \varkappa_{1111}^* \frac{\partial}{\partial y_1} u_1 + \varkappa_{1122}^* \frac{\partial}{\partial y_2} u_2 + \lambda_1 q_1 p \\ Q_{12}^* &= \varkappa_{1212}^* \frac{\partial}{\partial y_2} u_1 + \varkappa_{1221}^* \frac{\partial}{\partial y_1} u_2 \\ Q_{22}^* &= \varkappa_{2211}^* \frac{\partial}{\partial y_1} u_1 + \varkappa_{2222}^* \frac{\partial}{\partial y_2} u_2 + \lambda_2 q_2 p \\ Q_{21}^* &= \varkappa_{2112}^* \frac{\partial}{\partial y_2} u_1 + \varkappa_{2121}^* \frac{\partial}{\partial y_1} u_2 \end{aligned} \tag{1.6}$$

where p is a scalar quantity related to hydrostatic pressure. The expressions for the components of tensor $\varkappa^*$ and quantity q_n for various formulations of the problem appear in /11,14,15/; selection of these expressions determines the problem. In the theory of large (finite) initial deformations we have $q_n = \lambda_n^{-1}$. The components of tensor $\varkappa^*$ satisfy the conditions of symmetry $\varkappa_{mn\alpha\beta}^* = \varkappa_{\beta\alpha mn}^*$ /14 and 15/.

Note that formulas (1.4)–(1.6) for elastic bodies with initial stresses have been formulated in Cartesian coordinates of the initial state, hence all boundary value problems for bodies with initial stresses defined in y_j coordinates are formulated by analogy with respective problems of the classic linear theory of elasticity. It is only necessary to take into account that stresses Q_{11}^* and Q_{12}^* appear when $y_1 = \text{const}$, while stresses Q_{22}^* and Q_{21}^* when $y_2 = \text{const}$. From formulas (1.4)–(1.6) follow the equations for the determination of displacements u_1 and u_2, and the scalar quantity p of the form

$$L_{nm}u_m = 0; \quad n, \ m = 1, \ 2, \ 3; \quad u_3 = p \tag{1.7}$$

Here the differential operators are defined by expressions similar to those in /14 and 15/. We represent the general solution of system (1.7), by analogy to static problems /11,12/, in the form

$$\begin{aligned} u_1 &= \lambda_1^{-1} q_1^{-1} \left(-\frac{\partial^2}{\partial y_1 \partial y_2} \chi^{(1)} + \frac{\partial^2}{\partial y_2^2} \chi^{(2)} \right); \\ u_2 &= \lambda_2^{-1} q_2^{-1} \left(\frac{\partial^2}{\partial y_1^2} \chi^{(1)} - \frac{\partial^2}{\partial y_1 \partial y_2} \chi^{(2)} \right) \\ p &= \lambda_1^{-2} q_1^{-2} \Big\{ [\varkappa_{1111}^* - \lambda_1 q_1 \lambda_2^{-1} q_2^{-1} (\varkappa_{1122}^* + \varkappa_{1212}^*)] \frac{\partial^2}{\partial y_1^2} + \\ &\quad \varkappa_{2112}^* \frac{\partial^2}{\partial y_2^2} - \rho \frac{\partial^2}{\partial t^2} \Big\} \frac{\partial}{\partial y_2} \chi^{(1)} + \lambda_2^{-2} q_2^{-2} \Big\{ \varkappa_{1221}^* \frac{\partial^2}{\partial y_1^2} + \\ &\quad [\varkappa_{2222}^* - \lambda_2 q_2 \lambda_1^{-1} q_1^{-1} (\varkappa_{1122}^* + \varkappa_{1212}^*)] \frac{\partial^2}{\partial y_2^2} - \rho \frac{\partial^2}{\partial t^2} \Big\} \frac{\partial}{\partial y_1} \chi^{(2)} \end{aligned} \tag{1.8}$$

Functions $\chi^{(j)}$ are solutions of the equation

$$\det \| L_{nm} \| \chi^{(j)} = 0; \quad n, \ m = 1, \ 2, \ 3; \ j = 1, \ 2 \tag{1.9}$$

Below, we consider, as in /2–4,6,9,18–20), the case when the input plane dynamic problem when passing to moving Cartesian coordinates η_j (1.2) admits the transformation to the steady state problem. Equation (1.9) in coordinates (1.2) can be represented in the form

$$\left(\frac{\partial^2}{\partial\eta_1^2}-\frac{1}{\nu_1}\frac{\partial^2}{\partial\eta_2^2}\right)\left(\frac{\partial^2}{\partial\eta_1^2}-\frac{1}{\nu_2}\frac{\partial^2}{\partial\eta_2^2}\right)\chi^{(j)}=0 \tag{1.10}$$

where ν_j are roots of the equation

$$\nu^2+2A\nu+A_1=0 \tag{1.11}$$

in which the coefficients are defined as follows:

$$\begin{aligned}2A\varkappa_{2112}^* &= \varkappa_{1111}^*-\rho v^2+\lambda_1^2q_1^2\lambda_2^{-2}q_2^{-2}\varkappa_{2222}^*-2\lambda_1q_1\lambda_2^{-1}q_2^{-1}(\varkappa_{1122}^*+\varkappa_{1212}^*)\\ A_1\varkappa_{2112}^* &= \lambda_1^2q_1^2\lambda_2^{-2}q_2^{-2}(\varkappa_{1221}^*-\rho v^2)\end{aligned} \tag{1.12}$$

In the moving system of coordinates we represent solution (1.8) in the form

$$\begin{aligned}u_1&=\lambda_1^{-1}q_1^{-1}\left(-\frac{\partial^2}{\partial\eta_1\partial\eta_2}\chi^{(1)}+\frac{\partial^2}{\partial\eta_2^2}\chi^{(2)}\right);\\ u_2&=\lambda_2^{-1}q_2^{-1}\left(\frac{\partial^2}{\partial\eta_1^2}\chi^{(1)}-\frac{\partial^2}{\partial\eta_1\partial\eta_2}\chi^{(2)}\right)\\ p&=\lambda_1^{-2}q_1^{-2}\left\{[\varkappa_{1111}^*-\rho v^2-\lambda_1q_1\lambda_2^{-1}q_2^{-1}(\varkappa_{1122}^*+\varkappa_{1212}^*)]\frac{\partial^2}{\partial\eta_1^2}+\right.\\ &\left.\varkappa_{2112}^*\frac{\partial}{\partial\eta_2^2}\right\}\frac{\partial}{\partial\eta_2}\chi^{(1)}+\lambda_2^{-2}q_2^{-2}\left\{(\varkappa_{1221}^*-\rho v^2)\frac{\partial^2}{\partial\eta_1^2}+\right.\\ &\left.[\varkappa_{2222}^*-\lambda_2q_2\lambda_1^{-1}q_1^{-1}(\varkappa_{1122}^*+\varkappa_{1212}^*)]\frac{\partial^2}{\partial\eta_2^2}\right\}\frac{\partial}{\partial\eta_1}\chi^{(2)}\end{aligned} \tag{1.13}$$

Note that the quantities that define initial stresses appear in components of tensor $\varkappa^*$, the elongation coefficients λ_j, and in the quantities q_j. Respective expressions for the quantities $\varkappa^*, \lambda_j, q_j$ are given in /11, 14,15/. Further construction of solutions depends on properties of the roots of Eq.(1.11). It is shown in /10/ that the linearized problems considered here for elastic bodies with initial stresses do not reduce to respective problems of the classic linear theory of elasticity for an orthotropic body.

2. Complex potentials. Equations for static problems corresponding to (1.11) were considered in /10–12/. The uniqueness of solution of linearized problems was used in /20/, and the absence of inner stability loss in /13,18/. It is shown in /10 and 11/ that Eq.(1.11) in the case of static problems has no pure real positive roots. We assume that for fairly low (subsonic) velocities of motion that type of equations remains valid, hence Eq.(1.11) has no pure real positive roots also in the case of dynamic problems. This situation also occurs in the classic linear theory of elasticity /2–4,6,9,18–20/.

Consider the equation

$$\mu^4+2A\mu^2+A_1=0 \tag{2.1}$$

where A and A_1 are determined from formulas (1.12).

The above reasoning shows that Eq.(2.1) has no real roots, i.e. $\mathrm{Im}\,\mu_j\neq 0$.

We introduce the complex variables z_j $(j=1,2)$ defined by expressions

$$\begin{aligned}z_j&=y_1-vt+\mu_jy_2\equiv\eta_1+\mu_j\eta_2\\ \bar z_j&=y_1-vt+\bar\mu_jy_2\equiv\eta_1+\bar\mu_j\eta_2;\quad \mathrm{Im}\,\mu_j\neq0\end{aligned} \tag{2.2}$$

Taking into account (1.11), (2.1), and (2.2), we represent Eq.(1.10) in the form

$$\frac{\partial^4}{\partial z_1\partial\bar z_1\partial z_2\partial\bar z_2}\chi^{(j)}=0;\quad j=1,2 \tag{2.3}$$

Let us consider the introduction of complex potentials in the cases of unequal and equal roots of Eq.(2.1).

Unequal roots. We select the solution of Eq.(2.3) of the form

$$\chi^{(1)}=\lambda_1q_1\chi;\quad \chi^{(2)}=0;\ \chi=2\mathrm{Re}\,[F_1(z_1)+F_2(z_2)] \tag{2.4}$$

where $F_j(z_j)$ are analytic functions of complex variables z_j $(j=1,2)$.

We introduce new analytic functions $\Phi_j(z_j)$ defined by formulas

$$\begin{aligned}F_j''(z_j)&=\mu_jB_j^{-1}\Phi_j(z_j);\quad B_j=\varkappa_{1212}^*\mu_j^2-\lambda_1q_1\lambda_2^{-1}q_2^{-1}(\varkappa_{1221}^*-\rho v^2)\equiv\\ &\lambda_1^{-1}q_1^{-1}\lambda_2q_2[\varkappa_{2112}^*\mu_j^2+(\varkappa_{1111}^*-\rho v^2)+\lambda_1^2q_1^2\lambda_2^{-2}q_2^{-2}\varkappa_{2222}^*-\\ &\lambda_1q_1\lambda_2^{-1}q_2^{-1}(2\varkappa_{1122}^*+\varkappa_{1212}^*)]\mu_j^2,\quad j=1,2\end{aligned} \tag{2.5}$$

Equation (2.1) with allowance for (1.12) was used in the derivation of expression for the quantity B_j in (2.5). From (2.4), (2.5), (1.13), and (1.6) with allowance for (1.12) and (2.1) for the determination of stresses and displacements in terms of complex potentials $\Phi_j(z_j)$ we obtain the expressions

$$\begin{aligned}
&Q_{22}^* = 2\,\mathrm{Re}\,[\Phi_1'(z_1) + \Phi_2'(z_2)];\\
&\overline{Q_{21}}^* = -2\mathrm{Re}\,[\mu_1\gamma_{21}^{(1)}\Phi_1'(z_1) + \mu_2\gamma_{21}^{(2)}\Phi_2'(z_2)]\\
&Q_{12}^* = -2\,\mathrm{Re}\,[\mu_1\gamma_{12}^{(1)}\Phi_1'(z_1) + \mu_2\gamma_{12}^{(2)}\Phi_2'(z_2)]\\
&Q_{11}^* = 2\,\mathrm{Re}\,[\mu_1{}^2\gamma_{11}^{(1)}\Phi_1'(z_1) + \mu_2{}^2\gamma_{11}^{(2)}\Phi_2'(z_2)];\\
&u_k = 2\,\mathrm{Re}\,[\gamma_k^{(1)}\Phi_1(z_1) + \gamma_k^{(2)}\Phi_2(z_2)]\\
&\gamma_{21}^{(j)} = (\varkappa_{2112}^*\mu_j{}^2 - \lambda_1 q_1\lambda_2^{-1}q_2^{-1}\varkappa_{1212}^*)\,B_j^{(-1)}; \quad \gamma_1^{(j)} = -\mu_j{}^2 B_j{}^{-1}\\
&\gamma_{12}^{(j)} = (\varkappa_{1212}^*\mu_j{}^2 - \varkappa_{1221}^*\lambda_1 q_1\lambda_2^{-1}q_2^{-1})\,B_j^{-1}; \quad \gamma_2^{(j)} = \lambda_1 q_1\lambda_2^{-1}q_2^{-1}\mu_j B_j^{-1}\\
&\gamma_{11}^{(j)} = (\varkappa_{2112}^*\mu_j{}^2 - \lambda_1 q_1\lambda_2^{-1}q_2^{-1}\varkappa_{1212}^* - \rho v^2)\,B_j^{-1}
\end{aligned} \tag{2.6}$$

Thus formulas (2.6) define stresses and displacements in terms of analytic functions $\Phi_j(z_j)$ of the complex variables z_j in the case of uneven roots. The complex potentials $\Phi_j(z_j)$ represent a generalization of complex potentials of Lekhnitskii /16/ in the case of static problems of classic linear theory of elasticity of an orthotropic body.

Equal roots. Since in this case $\bar{\mu}_1 = -\mu_1$, we can represent the solution of Eq.(2.3) in the form

$$\begin{gathered}
\chi^{(j)} = \mathrm{Re}\,[F_1^{(j)}(z_1) + \bar{z}_1 F_2^{(j)}(z_1)]\\
z_1 = y_1 - vt + \mu_1 y_2 \equiv \eta_1 + \mu_1 y_2; \quad \bar{z}_1 = y_1 - vt - \mu_1 y_1 \equiv \eta_1 - \mu_1 y_2
\end{gathered} \tag{2.7}$$

where $F_n^{(j)}(z_1)$ are analytic functions of the complex variable z_1.

Below, we introduce new analytic functions $\varphi_j(z_1)$ of the complex variable z_1 defined by formulas

$$\begin{gathered}
F_j^{(1)\prime}(z_1) = {}^1/_2\mu_1 B^{-1}\varphi_j(z_1); \quad F_j^{(2)\prime}(z_1) = {}^1/_2 B^{-1}\varphi_j(z_1)\\
B = \varkappa_{1212}^*\lambda_1^{-1}q_1^{-1}\mu_1{}^2 - \lambda_2^{-1}q_2^{-1}(\varkappa_{1221}^* - \rho v^2) \equiv \mu_1{}^2\,[\varkappa_{2112}^*\mu_1{}^2\lambda_1^{-2}q_1^{-2}\lambda_2 q_2 +\\
\lambda_1^{-2}q_1^{-2}\lambda_2 q_2(\varkappa_{1111}^* - \rho v^2) + \lambda_2^{-1}q_2^{-1}\varkappa_{2222}^* - \lambda_1^{-1}q_1^{-1}(2\varkappa_{1122}^* + \varkappa_{1212}^*)]
\end{gathered} \tag{2.8}$$

Relations (2.1) and (1.12) have been taken into account in the last expressions of (2.8).

Substituting (2.7) and (2.8) into (1.13) and (1.6) and taking into account (1.12) and (2.1), for the determination of stresses and displacements in terms of complex potentials $\varphi_j(z_1)$ we obtain formulas

$$\begin{aligned}
&Q_{22}^* = \mathrm{Re}\,[\varphi_0(z_1) + \gamma_{22}^{(1)}\varphi_1'(z_1)];\\
&Q_{21}^* = \mathrm{Re}\,[\mu_1\gamma_{21}^{(1)}\varphi_0(z_1) + \gamma_{21}^{(2)}\varphi_2'(z_1)]\\
&Q_{12}^* = \mathrm{Re}\,[-\mu_1\gamma_{12}^{(1)}\varphi_0(z_1) + \gamma_{12}^{(2)}\varphi_2'(z_1)];\\
&Q_{11}^* = \mathrm{Re}\,[\mu_1{}^2\gamma_{11}^{(1)}\varphi_0(z_1) + \gamma_{11}^{(2)}\varphi_2'(z_1)]\\
&u_k = \mathrm{Re}\,[\gamma_k^{(1)}\varphi_0(z_1) + \gamma_k^{(2)}\varphi_2(z_1)]; \quad \varphi_0(z_1) = \varphi_1''(z_1) + \bar{z}_1\varphi_2''(z_1);\\
&k = 1, 2\\
&\gamma_{22}^{(2)} = -2\,[\lambda_1^{-2}q_1^{-2}\lambda_2 q_2\varkappa_{2112}^*\mu_1{}^4 + \lambda_2^{-1}q_2^{-1}(\varkappa_{1221}^* - \rho v^2)]\,B^{-1}\\
&\gamma_{21}^{(1)} = (\lambda_2^{-1}q_2^{-1}\varkappa_{1212}^* - \lambda_1^{-1}q_1^{-1}\varkappa_{2112}^*\mu_1{}^2)\,B^{-1}\\
&\gamma_{21}^{(2)} = 2\mu_1(\lambda_1^{-1}q_1^{-1}\varkappa_{2112}^*\mu_1{}^2 + \lambda_2^{-1}q_2^{-1}\varkappa_{1212}^*)\,B^{-1}\\
&\gamma_{12}^{(1)} = (\varkappa_{1212}^*\lambda_1^{-1}q_1^{-}\,\mu_1{}^2 - \varkappa_{1221}^*\lambda_2^{-1}q_2^{-1})\,B^{-1}\\
&\gamma_{12}^{(2)} = 2\mu_1(\varkappa_{1212}^*\lambda_1^{-1}q_1^{-1}\mu_1{}^2 + \varkappa_{1221}^*\lambda_2^{-1}q_2^{-1})\,B^{-1}\\
&\gamma_{11}^{(1)} = (\varkappa_{2112}^*\lambda_1^{-1}q_1^{-1}\mu_1{}^2 - \lambda_2^{-1}q_2^{-1}\varkappa_{1212}^* - \lambda_1^{-1}q_1^{-1}\rho v^2)\,B^{-1}\\
&\gamma_{11}^{(2)} = -2\,[\lambda_1^{-1}q_1^{-1}\varkappa_{2112}^*\mu_1{}^4 + \lambda_1 q_1\lambda_2^{-2}q_2^{-2}(\varkappa_{1221}^* - \rho v^2)]\,B^{-1}\\
&\gamma_1^{(1)} = -\gamma_1^{(2)} = -\lambda_1^{-1}q_1^{-1}\mu_1{}^2 B^{-1}; \quad \gamma_2^{(1)} = \gamma_1^{(2)} = \lambda_2 q_2\mu_1 B^{-1}
\end{aligned} \tag{2.9}$$

Thus in the case of equal roots formula (2.9) defines stresses and displacements in terms of analytic functions $\varphi_j(z_1)$ of the complex variable z_1. The complex potentials $\varphi_j(z_1)$ are generalizations of the Kolosov–Muskhelishvili complex potentials for static problems of the linear theory of elasticity of the isotropic body /17/.

Formulas (2.9) can be represented in a form close to that of /17/ by introducing the new functions

$$\varphi_1'(z_1) = \psi(z_1); \quad \varphi_2'(z_1) = \varphi(z_1); \quad \Phi(z_1) = \varphi'(z_1); \quad \Psi(z_1) = \psi'(z_1) \tag{2.10}$$

Using (2.10) we represent formulas (2.9) as

$$\begin{aligned} &Q_{22}^* = \operatorname{Re}[\Psi_0(z_1) + \gamma_{22}^{(1)}\Phi(z_1)]; \\ &Q_{21}^* = \operatorname{Re}[\mu_1\gamma_{21}^{(1)}\Psi_0(z_1) + \gamma_{21}^{(1)}\Phi(z_1)] \\ &Q_{12}^* = \operatorname{Re}[-\mu_1\gamma_{12}^{(1)}\Psi_0(z_1) + \gamma_{12}^{(2)}\Phi(z_1)]; \quad Q_{11}^* = \operatorname{Re}[\mu_1^2\gamma_{11}^{(1)}\Psi_0(z_1) + \\ &\quad \gamma_{11}^{(2)}\Phi(z_1)]; \quad u_k = \operatorname{Re}\{\gamma_k^{(1)}[\psi(z_1) + \bar{z}_1\varphi'(z_1)] + \gamma_k^{(2)}\varphi(z_1)\}, \quad k = 1, 2 \\ &\Psi_0(z_1) = \Psi(z_1) + \bar{z}_1\Phi'(z_1) \end{aligned} \tag{2.11}$$

Using the representations in terms of complex potentials of form (2.6) in the case of unequal roots and (2.9) in that of equal roots, we can obtain exact solutions of the class of problems considered in /2–4, 5–9/ and other publications of the classic linear theory of elasticity.

3. Method of solution. Values of complex parameters. Passage to the limit.

We shall indicate the method of constructing exact solutions of dynamic problems for bodies with initial stresses by using complex potentials obtained for respective static problems. First, it should be pointed out that a comparison of respective formulas for dynamic problems in the case of compressible bodies /1/ with formulas derived above for incompressible bodies shows that the expressions for stresses and displacements in terms of complex potentials have the same structure. The only difference appears in formulas defining the coefficients $\gamma_{nm}^{(i)}$ and $\gamma_k^{(j)}$. It is, thus, possible to obtain a solution of a general form common for both compressible and incompressible bodies.

Let us compare the expressions for stresses and displacements in terms of complex potentials for dynamic problems in the case of unequal roots (2.6) and in that of equal roots (2.9) with the respective expressions in /10 and 11/ for unequal and in /12/ for equal roots. The comparison shows that stresses Q_{22}^*, Q_{21}^* and Q_{11}^*, and displacements u_1 and u_2 are defined in terms of complex potentials by the same formulas, but the expressions defining stress Q_{12}^* differ. Note that in the case of static problems for elastic bodies with initial stresses exact solutions were obtained for the following plane problems. For the first, second and mixed half-plane problems in /10–12/, for the contact problem of the half-plane and a stamp free of friction in /12,21/; for the contact problem for the half-plane with friction at the stamp in /22/; for problems of normal cleavage crack and transverse and longitudinal shear under arbitrary loading of crack edges in /23–25/, and for problems of splitting of elastic bodies by a rigid wedge in /26/. These problems were reduced to problems for the lower half-plane at whose boundary $y_2 = 0$ particular combinations of quantities Q_{22}^*, Q_{21}^*, u_1 and u_2 were specified. As indicated above quantities Q_{22}^*, Q_{21}^*, u_1 and u_2 are expressed in terms of complex potentials by formulas that are of the same form for static and dynamic problems (the coefficients in these formulas are, certainly, defined in terms of known quantities by various formulas). The complex potentials for static problems, calculated on the basis of the exact solution, remain unchanged also for the respective static problems /10–12, 21–26/. Formulas for the determination of stresses Q_{22}^*, Q_{21}^* and Q_{11}^*, and of displacements u_1 and u_2 also remain unchanged. It is necessary to use the formulas obtained above only for the determination of stresses Q_{12}^*.

It is, thus, possible to formulate the following method of investigation (derivation of exact solutions) of dynamic problems for bodies with initial stresses that correspond to static problems for bodies with initial stresses /10–12/, 21–26/. It is necessary to introduce in the complex potentials /10–12, 21–26/ the complex parameters μ_j, the complex variables z_j, and coefficients $\gamma_{nm}^{(j)}$ and $\gamma_k^{(j)}$ (except $\gamma_{12}^{(1)}$ and $\gamma_{12}^{(2)}$) taken from this paper or from /1/. The complex potentials obtained in this way provide exact solutions of the respective dynamic problems (in the considered here formulation) of general form common to compressible and incompressible bodies. By a similar substitution we obtain from /10–12, 21–26/ also formulas for the determination of stresses Q_{22}^*, Q_{21}^* and Q_{11}^*, and displacements u_1 and u_2 for dynamic problems. For determining stress Q_{12}^* it is necessary to use formulas derived in this paper together with the complex potentials obtained by the method indicated above.

For the subsequent investigation of the effect of initial stresses on the considered dynamic processes it is necessary to determine the complex parameters μ_j and coefficients $\gamma_{nm}^{(j)}$ and $\gamma_k^{(j)}$ for bodies with elastic potentials of specific form. For this the formulas of monographs /13–15/, of the present paper and those in /1,10,11/ should be used. As an example, these quantities are calculated for incompressible bodies with elastic potentials of specific form.

Determination of complex parameters. For the Treloar potential (body of the neo-Hookean type) we obtain in conformity with /11,14,15/ and with allowance for (1.12), after a number of transformations, expressions for the coefficients in Eq.(2.1) of the form

$$2A = \lambda_1^2\lambda_2^{-2}(1-\varepsilon_2^2\lambda_1^{-2})+1; \qquad A_1 = \lambda_1^2\lambda_2^{-2}(1-\varepsilon_2^2\lambda_1^{-2}) \tag{3.1}$$
$$\varepsilon_2 = vc_s^{\circ -1}; \qquad c_s^{\circ 2} = 2C_{10}/\rho; \qquad c_{sy_2} = \lambda c_s^{\circ}$$

where c_s° is the shear wave velocity in the body free of load, c_{sy_2} is the shear wave velocity polarized in the y_1Oy_2 plane propagating along the Oy_1 axis in a body with initial stresses /27/, and C_{10} is a constant appearing as a multiplier in the expression for the elastic potential.

From (3.1) and (2.1) we obtain for the complex parameters expressions of the form

$$\mu_1 = i; \qquad \mu_2 = i\lambda_1\lambda_2^{-1}\sqrt{1-\varepsilon_2^2\lambda_1^{-2}} \equiv i\lambda_1\lambda_2^{-1}\sqrt{1-v^2c_{sy_2}^{-2}} \tag{3.2}$$

Formulas (3.1) and (3.2) were obtained in the theory of large (finite) initial deformations.

In the theory of small initial deformations /13–15/ for the linearly elastic body, when the initial condition is dertermined by the geometrically linear theory, the complex parameters are of the form

$$\mu_1 = i; \qquad \mu_2 = i\left[\left(1+\frac{\sigma_{11}{}^0}{\mu}-\varepsilon_2^2\right)\left(1+\frac{\sigma_{22}{}^0}{\mu}\right)^{-1}\right]^{1/2}; \quad \varepsilon_2 = vc_s^{-1}; \ c_s^2 = \frac{\mu}{\rho}$$

Complex parameters of bodies with elastic potentials of different structure can be determined in a similar form.

Passing to limit for equal roots. Setting in formulas (2.8) and (2.9) $v=0$ we obtain $\gamma_{12}^{(1)}=1$; $\gamma_{11}^{(1)}=-\gamma_{21}^{(1)}$, with all expressions in (2.8) and (2.9) turning into the respective expressions for static problems of bodies with initial stresses /12/. If, in addition, initial stresses are assumed zero, then, as shown in /12/, we obtain the Kolosov–Muskhelishvili formulas for static problems of the classic theory of elasticity for isotropic incompressible bodies.

Passing to the limit for unequal roots. Setting $v=0$ in formulas (2.6) we obtain $\gamma_{12}^{(j)}=1$; $\gamma_{12}^{(j)}=\gamma_{21}^{(j)}$; $j=1,2$. In that case all expressions in (2.5) and (2.6) become the respective expressions /11/ for static problems for incompressible bodies with initial stresses. If in addition initial stresses are assumed zero, then, as shown in /10/, we obtain Lekhnitskii's expressions for static problems of the classic theory of elasticity for orthotropic incompressible bodies /16/.

4. An example. The Rayleigh waves. As an example we shall consider the problem of propagation of Rayleigh surface waves in the half-plane $y_2<0$ with initial stresses. The velocity v is assumed unknown, and appropriate equations are derived for its determination. The analysis for equal and unequal roots is carried out separately. Equations for velocity determination are derived from the condition of existence in the half-plane of nonzero complex potentials, which ensure zero stresses at the half-plane boundary at $y_2=0$.

The case of equal roots. At the half-plane boundary at $y_2=0$ we obtain from (2.11) the boundary conditions

$$Q_{22}^* \equiv \operatorname{Re}\{[\Psi(z_1)+\bar z_1\Phi'(z_1)]+\gamma_{22}{}^{(1)}\Phi(z_1)\}=0 \tag{4.1}$$
$$Q_{21}^* \equiv \operatorname{Re}\{\mu_1\gamma_{21}{}^{(1)}[\Psi(z_1)+\bar z_1\Phi'(z_1)]+\gamma_{21}{}^{(2)}\Phi(z_1)\}=0$$

and from the condition of existence of nonzero solutions of system (4.1) we obtain for the determination of the Rayleigh wave velocity an equation of the form

$$\gamma_{21}{}^{(2)}-\mu_1\gamma_{21}{}^{(1)}\lambda_{22}{}^{(2)}=0$$

in which the quantities are determined using formulas (2.9) and (2.1).

The case of unequal roots. At the half-plane boundary we obtain from (2.6) the boundary conditions

$$Q_{22}^* \equiv 2\operatorname{Re}[\Phi_1'(z_1)+\Phi_2'(z_2)]=0 \tag{4.2}$$
$$Q_{21}^* \equiv -2\operatorname{Re}[\mu_1\gamma_{21}{}^{(1)}\Phi_1'(z_1)+\mu_2\gamma_{21}{}^{(2)}\Phi_2'(z_2)]=0$$

and from the condition of existence of nonzero solutions of system (4.2) we obtain for the determination of the Rayleigh wave velocity an equation of the form

$$\mu_2\gamma_{21}^{(2)} - \mu_1\gamma_{21}^{(1)} = 0 \tag{4.3}$$

in which the quantities are determined using formulas (2.1) and (2.6).

Numerical example. A body of the neo-Hookean type. In this case complex parameters are of the form (3.2). Since at $v < c_{sy_2}$ we have the case of unequal roots, hence the equation that defines the velocity of Rayleigh waves is of the form (4.3). Let us assume that the condition $S_{22} \equiv \sigma_{22}{}^{*0} = 0$ is satisfied at the half-plane.

From (2.3), (2.6), (3.1) and (4.3) we have

$$\mu_1\gamma_{21}^{(1)} - \mu_2\gamma_{11}^{(2)} = -\frac{i}{2}\frac{(x^3+x^2+3x-1)(x-1)}{x(1+x^2)}; \quad x = \frac{\lambda_1}{\lambda_2}\sqrt{1-v^2c_{sy_2}^{-2}} \tag{4.4}$$

When $v < c_{sy_2}$ we obtain from (4.3) and (4.4) the single equation

$$x^3 + x^2 + 3x - 1 = 0 \tag{4.5}$$

which coincides with the respective equations in /19/. Denoting by x_* the positive real root of Eq.(4.5) we obtain from the second expression of formulas (4.4) the relation

$$v^2 \equiv c_R^2 = c_{sy_2}^2(1 - x_*^2\lambda_2^2\lambda_1^{-2}) = \lambda_1^2 c_s\ (1 - x_*^2\lambda_2^2\lambda_1^{-2}) \tag{4.6}$$

If the initial state is also defined for the plane deformation $(\lambda_3 = 1)$, from (4.6) with (1.3) taken into account, we obtain

$$v^2 \equiv c_R^2 = c_{sy_2}^2(1 - x_*^2\lambda_1^{-4}) \tag{4.7}$$

The results (4.6) and (4.7) coincide with those in /19/.

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